Discrete Mathematics 27 (1979) 23–33.

INDEPENDENCE NUMBERS OF GRAPHS – AN EXTENSION OF THE KOENIG–EGERVARY THEOREM

Robert W. DEMING

Department of Mathematics, SUNY College at Oswego, Oswego, NY 13126, USA

Received 26 April 1976
Revised 10 April 1978

Let G be an arbitrary finite, undirected graph with no loops nor multiple edges. In this paper the inequality $\beta_0 \leq n - \beta_1$ is investigated where β_0 is the independence number of G, n is the number of vertices, and β_1 is the cardinality of a maximum edge matching. The class of graphs for which equality holds is characterized. A polynomially-bounded algorithm is given which tests an arbitrary graph G for equality, and computes a maximum independent set of vertices when equality holds. Equality is "prevented" by the existence of a blossom-pair – a subgraph generated by a certain subset m_i of edges from a maximum edge matching M for G. It is shown that $\beta_0 = n - \beta_1 - |R|$ where R is a minimum set of representatives of the family $\{m_i\}$ of blossom pair-generating subsets of M. Finally, apolynomially-bonded algorithm is given which partitions an arbitrary graph G into subgraphs $G_0, G_1, \ldots, G_q$ such that $\beta_0(G) = \sum_{i=0}^{q} \beta_0(G_i)$. Moreover, if arbitrary maximum independent subsets of vertices $S_1, S_2, \ldots, S_q$ are known, then a polynomially-bounded algorithm computes a maximum independent set S_0 for G_0 such that $S = \bigcup \{S_i; i = 0, 1, \ldots, q\}$ is a maximum independent subset for G.

1. Introduction

Graphs considered here are assumed to be finite, undirected, with no loops, and with no multiple edges. The number of vertices is denoted by n, the cardinality of a largest set of independent vertices by β_0, and the cardinality of a largest edge matching is denoted by β_1. Trivially

$$\beta_0 \leq n - \beta_1 \tag{1}$$

for an arbitrary graph G, and it is well-known that equality holds in (1) for bipartite graphs (Koenig–Egervary theorem). In Section 2, the class of all graphs for which equality holds in (1) is characterized. In addition an algorithm, which when used with Edmond's edge-matching algorithm [1], gives a polynomially-bounded algorithm which produces a maximum independent set of vertices, or indicates that equality does not hold in (1). For convenience, graphs which satisfy equality in (1) will be called *Koenig-Egervary graphs*, or simply *K–E graphs* in this paper.

Let M be a maximum edge matching in an arbitrary graph G. Equality in (1) is prevented by the existence of certain subgraphs B_i (blossom pairs) defined in Section 2. These subgraphs are generated by subsets m_i of M. Let $\mathcal{M} = \{m_i\}$ be the family of all blossom pair-generating subsets of M. It is shown in Section 4 that $(n-\beta_1)-\beta_0$ is the cardinality of a minimum set of representatives of M; i.e., a subset $r \subset M$ of minimum size for which $r \cap m_i \neq \emptyset$ for all $m_i \in \mathcal{M}$. Moreover; if such a set r is known, the problem of finding a maximum independent set of vertices for G can be solved thereon by a polynomially-bounded algorithm. Using the results of Section 4, a polynomially-bounded algorithm is developed in Section 5 which partitions an arbitrary graph G into subgraphs $G_0, \ldots, G_q$ such that $\beta_0(G)=\sum_{i=0}^{q}\beta_0(G_i)$. In addition, once maximum independent sets S_i of vertices for $G_1, \ldots, G_q$ are known, a maximum independent set of vertices for G can be computed efficiently.

2. K–E Graphs

Let G be an arbitrary graph and let M be a maximum edge-matching for G. The edges of M will be called *heavy*; those not in M will be called *light*. An *alternating* path from vertex x to vertex y is a path $p(x, y)$ whose edges are alternately light and heavy. An alternating path may begin (or end) with either a light or heavy edge. A vertex x is *exposed* relative to M (or *not spanned* by M) if x is not the endpoint of a heavy edge. An odd cycle $u_0, u_1, \ldots, u_{2k}, u_0, k \geq 1$, is called a *blossom* if edges $u_1u_2, u_3u_4, \ldots, u_{2k-1}u_{2k}$ are heavy. Vertex u_0 is called the *blossom tip*. (This terminology is that of Edmonds [1]).

If S is a maximum independent set of vertices in G such that

$$|S| = n - |M|, \tag{2}$$

then

(a) S necessarily contains every exposed vertex, and

(b) S contains exactly one endpoint of each heavy edge.

In view of (b), S contains exactly one of the vertices u_1 or u_{2k} in a blossom $u_0, u_1, \ldots, u_{2k}, u_0$; hence *$S$ does not contain a blossom tip*. One may conclude then that (2) cannot be satisfied by a maximum independent set S if

> There is an alternating path $u_1, \ldots, u_{2k-1}$ $(k \geq 1)$ where $u_2u_3, \ldots, u_{2k-2}u_{2k-1}$ are heavy edges and u_1 is exposed, u_{2k-1} is a blossom tip. (Note: for $k=1$, u_1 is both exposed and a blossom tip.) (3)

or if;

> there is an alternating path $u_1, u_2, \ldots, u_{2k}$ $(k \geq 1)$ where u_1u_2 and $u_{2k-1}u_{2k}$ are heavy and u_1 and u_{2k} are blossom tips. (4)

As it turns out, if neither of configurations (3) or (4) is present then Algorithm A produces a maximum independent set of vertices S which satisfies (2). (The author is grateful for the referee's suggestions which greatly simplifies this algorithm and subsequent results in this paper.) It is assumed that a maximum edge matching M for G has been computed.

Algorithm A.

Step 0. Let $S = \emptyset$ and $H = G$.

Step 1. If $H = \emptyset$, stop. S is the desired independent set. If H contains an exposed vertex z, color z red and set FLAG = 1. If there are no exposed vertices, select a heavy edge in H, color one of its vertices red, the other blue, and set FLAG = 0.

Step 2. If no red vertex is adjacent to an uncolored vertex, place all red vertices in S, delete all colored vertices from H, and return to Step 1. If red vertex u is adjacent to uncolored vertex v, then edge uv is light and there is a heavy edge vw where w is necessarily uncolored also. (If there is no heavy edge vw, v is exposed and there is an odd alternating path $z, u_1, \ldots, u_{2k}, v$ with edges $u_1u_2, \ldots, u_{2k-1}u_{2k}$ heavy. Since z is exposed, this contradicts the fact that M was a maximum edge matching.) Color v blue and w red. Define a "predecessor function" p by $p(v) = u$ and $p(w) = v$.

Step 3. If no two red vertices are adjacent, return to Step 2. If two red vertices are adjacent, a blossom has been found whose tip x is pointed out by the predecessor function p. If FLAG = 0; x is an endpoint of a heavy edge xy; erase all colors, color x blue, y red, set FLAG = 2, and return to Step 2. If FLAG = 1; the predecessor function points out a path from the blossom tip x and the exposed vertex z. Stop; by virtue of (3), the desired independent set S does not exist. If FLAG = 2; the predecessor function points out a path between two blossom tips. Stop; by virtue of (4), the desired independent set S does not exist.

Algorithm A terminates when

(a) H is empty in case the desired independent set S has been found, or

(b) when a configuration of type (3) or (4) is found in which case the desired independent set S does not exist.

Thus K–E graphs can be characterized as those graphs free of configurations (3) and (4) relative to a maximum edge matching M. Note that existence of configurations (3) and (4) does not depend on the choice of maximum matchings since β_0 cannot simultaneously be both equal to and strictly less than $n - \beta_1$.

If G admits a perfect matching M (no exposed vertices) then G is K–E iff no configuration of type (4) is present. An arbitrary graph G with maximum edge matching M can be extended to a graph G' with the properties

(a) G' admits a perfect matching $M' \supseteq M$.

(b) G' is K–E iff G is K–E.

(c) $\beta_0(G') = \beta_0(G)$; moreover a maximum independent set S for G can be obtained from a maximum independent set S' for G' by a simple replacement operation.

To construct G', let X be the set of exposed vertices of G relative to a maximum matching M. For each $x \in X$, add a new vertex x' and new edges xx' and $\{x'y : y$ is adjacent to $x\}$. $M' = M \cup \{xx' : x \in X\}$ is a perfect matching for G'. If S' is a maximum independent set of vertices in G', replace each "new" vertex $x' \in S'$ by its associate $x \in X$. The resulting set S is independent in G and $|S| = |S'|$. Since $\beta_0(G) \leq \beta_0(G')$, $|S| = |S'|$ implies that $\beta_0(G) = \beta_0(G')$. Furthermore; $n(G') - \beta_1(G') = n(G) - \beta_1(G)$. Thus G' is K–E iff G if K–E.

A configuration of type (4) will be called a *blossom pair*. Thus, if G admits a perfect matching M then G is K–E iff G contains no blossom pair relative to M. More generally

Theorem 1. *An arbitrary graph G is* K–E$\Leftrightarrow$*for any maximum matching M, the extension G' with perfect matching M' contains no blossom pair.*

An alternate characterization of K–E graphs is useful for further analysis of the inequality (1) $\beta_0 \leq n - \beta_1$. Let M be a perfect matching for G. (In view of the preceding paragraph, there is no loss of generality in assuming M to be a perfect matching.) Let x_1x_2 and y_1y_2 be heavy edges belonging to a blossom pair B. Then for each $i, j = 1, 2$, there is an alternating path $p(x_i, y_j)$ beginning and ending with light edges and contained in B. Conversely; two heavy edges x_1x_2 and y_1y_2 of G will be called *biconnected* if there are alternating paths $p(x_i, y_j)$ for $i = 1, 2$; $j = 1, 2$ beginning and ending with light edges. A straightforward argument shows that a pair of biconnected heavy edges x_1x_2 and y_1y_2 are contained in a blossom pair B. Thus

Theorem 2. *An arbitrary graph G is* K–E$\Leftrightarrow$*for any maximum matching M, the extension G' contains no pair of biconnected heavy edges.*

We conclude with several examples.

Example 1. View Fig. 1. In (a) K_4 with a perfect matching is a blossom pair with blossom tips u_0 and w_0. On the other hand, a graph may contain K_4 and yet be

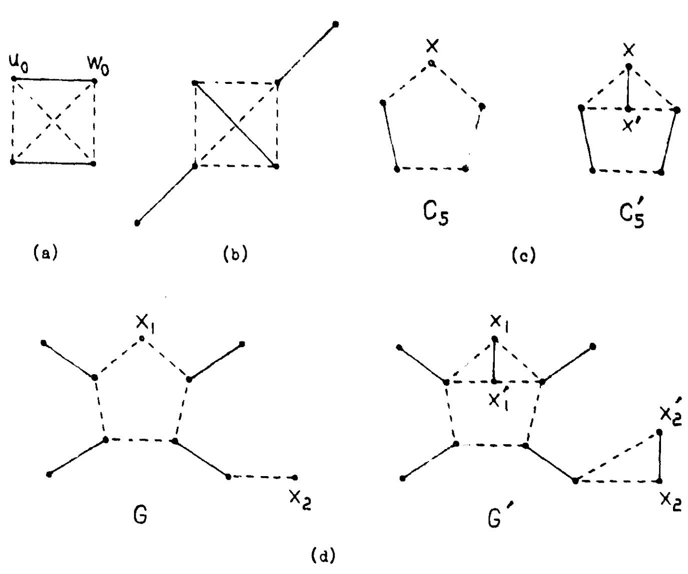

Fig. 1

K–E (b). (c) and (d) illustrate the extension of a graph to a graph with a perfect matching, one of which is not K–E (c) while the other is K–E (d).

3. An application to set covering and graph coloring

Let X be a finite set and $A=\{A_1,\dots,A_m\}$ be a family of subsets of X which covers X. Let $G(X,A)$ be the graph with vertices X and edges x_ix_j if $i\neq j$ and x_i and x_j both belong to some $A_k\in A$. A straightforward argument shows that l, the cardinality of a minimum subfamily of A which covers X satisfies

$$\beta_0(G)\leq l\leq n(G)-\beta_1(G). \tag{5}$$

If G is K–E, then $l=n(G)-\beta_1(G)$. The latter number is the cardinality of a minimum edge cover (of vertices) of G which can be obtained from a maximum matching by well-known methods. If $x_1x_2,\dots,x_{2l-1}x_{2l}$ is a minimum edge cover, choose $A_{i_1},\dots,A_{i_l}$ from A such that A_{i_j} contains both x_j and x_{j+1}. This gives a minimum cover of X.

As a special case, let X be the vertices of a graph G and let $\mathcal{S}=\{S_1,\dots,S_m\}$ be the family of all independent sets of vertices of G, or, equivalently, $\mathcal{S}$ is the family of all cliques in the graph-theoretic complement G^* of G. The cardinality of a minimum cover of X by members of $\mathcal{S}$ is the chromatic number of G. Moreover, the graph $G(X,\mathcal{S})$ is precisely G^*. Applying (5)

$$\beta_0(G^*)\leq\chi(G)\leq n(G^*)-\beta_1(G^*) \tag{6}$$

where $\chi(G)$ is the chromatic number of G. If G^* is K–E, a minimum edge cover of G^* determines a minimum coloring for G.

4. An identity for arbitrary graphs

Let M be a perfect matching for G and let $b=\{b_1,\dots,b_q\}$ where $b_i=B_i\cap M$ and B_i is a blossom pair in G relative to M; that is, each b_i consists of all heavy edges belonging to some blossom pair B_i. A subset $R=\{e_1,\dots,e_p\}$ of heavy edges of M is a *minimum set of representatives* of b if (1) $R\cap b_i$ is not empty for each $i=1,\dots,q$, and (2) no subset of M with smaller cardinality than R has property (1). We abbreviate a minimum set of representatives as *an m.s.r.* The basic result of this section is

Theorem 3. *If $R=\{e_1,\dots,e_p\}$ is an m.s.r. for the set b, then $\beta_0(G)=n(G)-\beta_1(G)-|R|$.*

We require two lemmas to prove Theorem 3.

Lemma 1. *If $\beta_0(G)>n(G)-\beta_1(G)-|R|$, then every maximum independent subset S contains an endpoint of at least one edge in R.*

Proof. Suppose not. Delete the vertices spanned by the edges of R from G. Let G_1 be the resulting subgraph. Our assumption that there is an S which does not contain an endpoint of an edge if R implies $\beta_0(G_1) = \beta_0(G)$. But G_1 is K–E; so

$$\beta_0(G) = \beta_0(G_1) = n(G_1) - \beta_1(G_1) = n(G) - \beta_1(G) - |R|,$$

a contradiction to $\beta_0(G) > (G) - \beta_1(G) - |R|$.

Lemma 2. $|R| \leqslant |M| - 1$ *for every graph G and; in addition,* $|R| = |M| - 1$ *iff G is a complete graph.*

Proof. If G is a complete graph, then any two heavy edges generate K_4 which is in this case, a blossom pair. Thus R must contain all but one edge of M. Suppose G is not complete. Let x and y be non-adjacent vertices which are endpoints of heavy edges e and e', respectively. Any blossom pair containing either e or e' must contain at least one other heavy edge $e'' \neq e$ or e'. Then $M \setminus \{e, e'\}$ is a set of representatives of b and the lemma follows.

Proof of Theorem 3. Let $e_i = x_i y_i$ for each $e_i \in R$. Delete the vertices x_i and y_i for $i = 1, \ldots, p$, along with incident edges from G. $M \setminus R$ is a perfect matching for the resulting graph G_1. Since R is an m.s.r. for b, G_1 contains no blossom pair, hence is K–E. Then $\beta_0(G) \geqslant \beta_0(G_1) = n(G) - \beta_1(G) - |R|$ and so

$$\beta_0(G) \geqslant n(G) - \beta_1(G) - |R| \tag{6}$$

For $n \geqslant 3$, let $\mathcal{G}_n$ be the class of all n-vertex graphs which admit perfect matchings and for which inequality (6) is strict. By Lemma 2 $K_n \notin \mathcal{G}_n$. Therefore, the class $\mathcal{G}_n$ contains a graph G which has a maximal number of edges. G admits a simple description: if S is a maximum independent set of vertices, then $G \setminus S$ is a complete subgraph, and sx is an edge of G for each $s \in S$ and for each $x \in G \setminus S$. This description of G must hold; for if not, an edge uv can be added to C where at least one of the vertices u or v is in $G \setminus S$. The resulting graph G_1 has the properties

(1) $\beta_0(G_1) = n(G_1) - \beta_1(G_1) - |R(G_1)|$ by maximality of G in $\mathcal{G}_n$,
(2) $|R(G)| \leqslant |R(G_1)|$ and
(3) $\beta_0(G) = \beta_0(G_1)$.

(1), (2), and (3) imply (along with the fact that obviously $\beta_1(G_1) = \beta_1(G)$);

$$\beta_0(G) = \beta_0(G_1) = n(G_1) - \beta_1(G_1) - |R(G_1)| \leqslant n(G) - \beta_1(G) - |R(G)|$$

a contradiction to $G \in \mathcal{G}_n$. With this description of G, the proof proceeds as follows. Let B be a blossom pair. Suppose every heavy edge in B has an endpoint in S. Consider the subgraph $G(B)$ generated by the vertices spanned by the heavy edges of B. Since $G(B) \cap (G \setminus S)$ is complete, $G(B) \cap S$ is a maximum independent set of $G(B)$ which consists of one endpoint of each edge of the perfect matching $M \cap G(B)$. This contradicts Theorem 1. Thus every blossom pair of G contains a heavy edge whose endpoints both lie in $G \setminus S$. Let G' be the subgraph of $G \setminus S$ generated by the vertices spanned by heavy edges of $G \setminus S$. G' is complete, so by Lemma 2, an m.s.r. R' must contain all but one heavy edge of G'. Thus an m.s.r. R for G must contain all but possibly one heavy edge e' of G'. On the other hand, Lemma 1 requires that R contain one heavy edge e which has one endpoint in S. Let R' be obtained by deleting e from R and replacing it by e'. Then $|R'| = |R|$ and R' is clearly a set of representatives of the family b of blossom pairs of G. Thus R' is an m.s.r. for family b. Since no edge in R' has a vertex in S, $\beta_0(G) \leq n(G) - \beta_1(G) - |R'|$ by Lemma 1. This inequality, in turn, contradicts $G \in \mathcal{G}_n$, and hence; implies $\mathcal{G}_n$ is empty, completing the proof.

A maximum independent set S for an arbitrary graph G with perfect matching M can be computed easily once the difficult problem of finding an m.s.r. R for the family b of blossom pairs has been solved. Let G_1 be the graph obtained by deleting the vertices x_i, y_i spanned by heavy edges $e_i \in R$, $i = 1, \ldots, p$ along with all incident edges from G. Then G_1 is K–E so

$$\beta_0(G_1) = n(G_1) - \beta_1(G_1) = (n(G) - 2p) - (\beta_1(G) - p)$$
$$= n(G) - \beta_1(G) - p = \beta_0(G).$$

That is to say; a maximum independent set of vertices S for G_1 (computed by Algorithm A) is a maximum independent set of vertices for G.

Example 2. View Fig. 2. $R = \{e_1, e_2\}$ is an m.s.r. and vertices $\{1, 5, 7, 9, 13\}$ form a maximum independent set for G_1, hence for G.

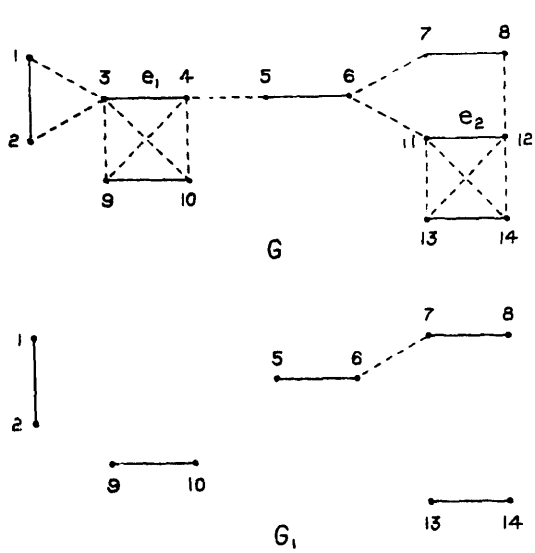

Fig. 2

5. A decomposition of arbitrary graphs

The relation "e is biconnected to e'" defined in Section 2 is an equivalence relation on the set M, a perfect matching for a graph G (or for its extension G' when G does not admit a perfect matching). Let $M_1, M_2, \ldots, M_q$ be the equivalence classes which contain more than one edge and let M_0 be the set of remaining edges. Let $G_0, G_1, \ldots, G_q$ be the subgraphs of G generated by the vertices spanned by the edges of $M_0, \ldots, M_q$, respectively. G_0 is K–E, and each blossom pair B is contained in exactly one of $G_1, \ldots, G_q$. (Theorem 2) Thus, if R_i is a minimum set of representatives of the set $\{B_i\}$ of blossom pairs contained in G_i for $i = 1, \ldots, q$, then $R = \bigcup \{R_i : i = 1, \ldots, q\}$ (disjoint union) is an m.s.r. for the set of blossom pairs in G. Moreover

$$\begin{aligned}\sum_{i=0}^{q} \beta_0(G_i) &= \sum_{i=0}^{q} (n(G_i) - \beta_1(G_i) - p(G_i)) \\ &= n(G) - \beta_1(G) - p(G) \\ &= \beta_0(G) \end{aligned} \tag{7}$$

where $p(G_i) = |R_i|$ and $p(G) = \sum |R_i|$. A stronger result is contained in Theorem 4, below. Certain properties of the decomposition $G_0, \ldots, G_q$ follow easily from the definition and Theorem 2.

> If x and y are vertices of G_i and G_j respectively where $i \neq j$ and $i, j \neq 0$, then there is no path $p(x, y)$ of the form $x, v_1, v_2, \ldots, v_{2k}, y$ $(k \geq 0)$ where $v_1v_2, \ldots, v_{2k-1}v_{2k}$ are heavy edges and the remaining edges are light. In particular no vertex of G_i is adjacent to a vertex of G_j when $i, j \neq 0$. (8)

> If x and y are vertices of G_i, $i \neq 0$, then there is no path $p(x, y)$ of the form $x, v_1, \ldots, v_{2k}, y$ $(k \geq 1)$ where $v_1v_2, \ldots, v_{2k-1}v_{2k}$ are heavy edges *in* G_0 and the remaining edges are light. (9)

(8) follows from the fact that x is the endpoint of a heavy edge xx' in G_i. Since $i \neq 0$, xx' is biconnected to a heavy edge uu' in G_i. In particular there are alternating light-heavy paths $p(x', u)$, $p(x', u')$ each beginning and ending with a light edge. Similarly, y is adjacent to y' via a heavy edge yy' in G_j and there are light-heavy alternating paths $p(y', w)$, $p(y', w')$ where ww' is a heavy edge of G_j. If the path $p(x, y)$ exists as described in (1), then uu' and ww' are biconnected by the paths and edges, $p(x', u)$, $p(x', u')$, xx', $p(x, y)$, $p(y', w)$, $p(y', w')$ and yy'. But M_i and M_j are disjoint by definition. (9) follows by a similar argument showing that if a path $p(x, y)$ has heavy edges $v_iv_2, \ldots, v_{2k-1}v_{2k}$ in G_0, then each is biconnected to a heavy edge yy' for instance, in G_i. This contradicts the fact that no edge of M_0 is biconnected to another edge. Thus no such path $p(x, y)$ exists.

Theorem 4. *Let S_i be any maximum independent set of vertices for G_i; $i=1,\ldots,q$. Then there is a maximum independent set S_0 of vertices for G_0 such that $S=\bigcup\{S_i : i=0,1,\ldots,q\}$ is a maximum independent set of vertices for G.*

Proof. For any S_0, sets $S_0,\ldots,S_q$ are disjoint; so $|S|=\sum_{i=0}^{q}|S_i|$. Since $|S_i|=\beta_0(G_i)$ and from (7), $\beta_0(G)=\sum_{i=0}^{q}\beta_0(G_i)$, we have $\beta_0(G)=|S|$. By (8), no vertex of S_i is adjacent to a vertex of S_j when $i\neq j$ and neither i nor j is 0. Thus, $S'=\bigcup\{S_i : i=1,\ldots,q\}$ is independent in G. It suffices to show that $S_0\subseteq G_0$ can be chosen so that $S'\cup S_0=S$ is independent in G. Let T be the set of vertices in G_0 which are adjacent to some vertex of S'. Delete the vertices of T, along with incident edges, from G_0. Let H be the resulting subgraph of G_0. The remaining set of heavy edges M_0' constitute an edge-matching for H. We show that M_0' is a maximum edge matching. Let x_1 and x_2 be vertices of H which are exposed relative to M_0'. Suppose there is an alternating path (augmenting path) $p(x_1,x_2)=x_1, v_1, v_2, \ldots, v_{2k}, x_2$, $k\geq 0$ in H where edges $v_1v_2,\ldots,v_{2k-1}v_{2k}$ are heavy. Since M_0 is a perfect matching for G_0, x_1 and x_2 are endpoints of heavy edges x_1y_1. x_2y_2 in M_0 where y_1, y_2 are vertices belonging to T. Let y_1 be adjacent to vertex $u_i\in S_i$ and y_2 be adjacent to $u_j\in S_j$ where $i,j\neq 0$. The resulting alternating path $p(u_i,u_j)=u_i, y_1, p(x_1,x_2), y_2, u_2$ contradicts property (8) if $i\neq j$, and (9) if $i=j$. Thus, the augmenting path $p(x_1,x_2)$ cannot exist and so M_0' is a maximum matching for H. Now, both G_0 and H are K–E, so

$$\beta_0(G_0)=n(G_0)-|M_0| \quad\text{and}\quad \beta_0(H)=n(H)-|M_0'|.$$

Since M_0 is a perfect matching for G_0, $|M_0'|=|M_0|-|T|$. Thus $n(G_0)-|M_0|=n(H)-|M_0'|$; that is, $\beta_0(G_0)=\beta_0(H)$. Thus, we may choose S_0 to be a maximum independent set of vertices in H and the last equality shows that S_0 will be a maximum independent set of vertices of G_0 also. Since no vertex of S_0 can be adjacent to a vertex of S_i, $i\neq 0$; $S=S'\cup S_0$ is independent in G, which completes the proof.

If $i\neq j$ and neither i nor j is 0 then no vertex of G_i is adjacent to a vertex of G_j by (8). The decomposition of G into the subgraphs $G_0, G_1,\ldots,G_q$ can therefore be obtained by calculating G_0 and removing it from G. The subgraphs $G_1,\ldots,G_q$ are then the usual connected components of $G\setminus G_0$. Algorithm B is a polynomially-bounded algorithm which computes G_0 – the subgraph generated by vertices spanned by all heavy edges of G which are not biconnected to another heavy edge relative to a perfect matching M for G. The set of heavy edges $M_0\subseteq M$ which generate G_0 are equivalently characterized as those heavy edges not belonging to a blossom pair. (See Theorem 2.) Let M be a perfect matching for G (or for the extension G' of G associated with a maximum matching for G when G does not admit a perfect matching as in Section 2).

Algorithm B.

Step 0. Set UNSCAN $=H=G$, $G_0=\emptyset$.

Step 1. If UNSCAN $=\emptyset$ stop. Otherwise set FLAG $=0$, choose a heavy edge uv in UNSCAN, and set $x_0=u$, $y_0=v$.

Step 2. Color x_0 red, y_0 blue.

Step 3. If no uncolored vertex of H is adjacent to a red vertex, go to Step 5.

Step 4. Choose an uncolored vertex $y \in H$ adjacent to a red vertex x (necessarily via a light edge). Then y is adjacent to an uncolored vertex w via a heavy edge yw. Color w red and y blue. Return to Step 3.

Step 5. If two red vertices are adjacent, go to Step 6. Otherwise, add the vertices x_0, y_0 to G_0, delete vertices x_0, y_0 from UNSCAN and from H, erase all colors and return to Step 1.

Step 6. If FLAG $= 0$, go to Step 7. Otherwise delete vertices x_0, y_0 from UNSCAN, erase all colors and return to Step 1.

Step 7. Set $x_0 = v$, $y_0 = u$, FLAG $= 1$, and return to Step 2.

The subgraph "G_0" produced by Algorithm B is the subgraph spanned by vertices of the edges of M_0. That is; a heavy edge uv is placed in G_0 by Algorithm B iff uv does not belong to a blossom pair. The reason is that uv belongs to a blossom pair iff there are alternating paths $u, x_1, \ldots, x_{2k}$ ($k \geq 1$) and $v, y_1, \ldots, y_{2l}$ where $x_1x_2, \ldots, x_{2k-1}x_{2k}$ and $y_1y_2, \ldots, y_{2l-1}y_{2l}$ are heavy edges and x_{2k} and y_{2l} are blossom tips (possibly the same blossom tip). When FLAG $= 0$ and $u = x_0$, x_{2k} is colored red. Subsequent coloring of the blossom containing x_{2k} leads to red vertices which are adjacent. Similarly, when FLAG $= 1$, and $x_0 = v$, y_{2l} is colored red with the subsequent coloring of the blossom containing y_{2l} giving adjacent red vertices. Thus $uv \notin M_0$ iff in both cases FLAG $= 0$ and FLAG $= 1$, adjacent vertices are colored red. But this is precisely the test used in Algorithm B to reject uv from membership in G_0. Finally deleting an edge uv once it is found to be in M_0 cannot affect later test for a heavy edge $u'v'$. If $u'v'$ belongs to a blossom pair of G then $u'v'$ belongs to a blossom pair of $G \setminus uv$. Otherwise, uv and $u'v'$ belong to a common blossom pair. But this contradicts the fact that $uv \in M_0$.

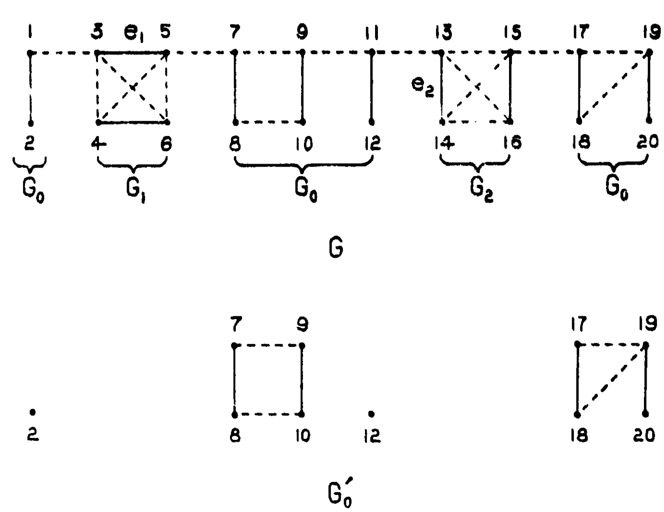

Fig. 3

Example 3. View Fig. 3. G_0' in this example is derived from the choice $S_1 = \{3\}$, $S_2 = \{13\}$. A maximum independent set in G_0', hence for G_0 is $S_0 = \{2, 7, 10, 12, 17, 20\}$. Then $S = \{2, 3, 7, 10, 12, 13, 17, 20\}$ is a maximum independent set for G. Note that $R = \{e_1, e_2\}$ for example is an m.s.r. for the family of blossom pairs. Thus $\beta_0 = 8$, $\beta_1 = 10$, $|R| = 2$ and $n = 20$. These numbers satisfy $\beta_0 = n - \beta_1 - |R|$.

6. Conclusion

Algorithms for computing independence numbers for arbitrary graphs have time bounds of $O(a^n)$ where $1 < a \leqslant 2$ and n is the number of vertices of the graph. The decomposition of an arbitrary graph G into subgraphs $G_0, G_1, \ldots, G_q$ obtained in Section 5 reduces this time bound

$$\text{from} \quad O(a^{\sum_{i=0}^{q} n_i}) \quad \text{to} \quad O\left(\sum_{i=1}^{q} a^{n_i}\right).$$

In attempting further reductions in bounds, attention can be confined to graphs of the type G_i for $i \geqslant 1$: graphs that admit a perfect matching and for which each heavy edge belongs to a blossom pair.

References

[1] J. Edmonds, Paths, trees, and flowers, Canad. J. Math. 17 (1965) 449–467.